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Necessary Condition and Genericity of Dynamic Feedback Linearization*

P. Rouchon

Abstract

A new necessary condition for dynamic feedback linearization in the sense of [3] is proposed. This condition concerns control systems $\dot{x} = f(x, u)$ with strictly less control variables than state variables. This necessary condition allows to prove the non-genericity of dynamic feedback linearizability, for the Whitney C^∞ topology on mappings $(x, u) \rightarrow f(x, u)$. However, this topology reveals to be too coarse to capture the nature of practical uncertainties: the polymerization reactor studied in [17] is shown to be linearizable via dynamic feedback for generic kinetic and thermal laws.

Key words: dynamic feedback linearization, flatness, elimination theory, structural stability, chemical reactor control

1 Introduction

In [18] the genericity and structural stability of affine control systems which are linearizable via dynamic feedback [3] is investigated. We address here a similar problem for general control systems where the dependence with respect to the control variables is not supposed to be affine. We prove that, generically (in the sense of the Whitney C^∞ topology for mappings $f(x, u)$), systems $\dot{x} = f(x, u)$ with $m \leq n - 1$ ($n = \dim(x)$ and $m = \dim(u)$) are not linearizable via dynamic feedback in the sense of [3]: the set of systems that are not linearizable via dynamic feedback contains an open and dense subset.

The demonstration of this somehow negative result is based on a new necessary dynamic feedback linearizability condition that generalizes to smooth systems and exogenous dynamic compensators the necessary flatness condition given in [7] (see [13] for a definition of endogenous and

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exogenous dynamic compensators). In [16] Sluis proposes, independently of us, the same necessary condition for smooth systems linearizable via a special class of dynamic compensators. We prove here that this necessary condition is general and does not rely on any particular subclass of dynamic compensators.

This non-genericity seems to be in contradiction with the fact that many practical systems are linearizable and flat (see, e.g., [4, 1, 6, 14]). In order to understand this apparent contradiction, we sketch an explanation via a representative case-study: the polymerization reactor considered in [17].

Although this paper addresses the C^∞ case, the results presented here have been strongly influenced by the differential-algebraic approach of dynamic feedback linearization and the notion of flatness [13, 6, 7, 5, 8]. In particular, the proof of the necessary condition is fundamentally based on the concept of linearizing (or flat) output due to Martin [13]. For another connected approach to the dynamic feedback linearization problem via Cartan's equivalence see also [15, 16].

The paper is organized as follows. In Section 2, the necessary condition for C^∞ systems is presented. In Section 3, we establish the second main result showing the non-genericity of linearizable systems. In Section 4, we discuss this result and its practical implications on the representative case-study of a polymerization reactor [17] that is shown, in the appendix, to be linearizable via dynamic feedback for generic kinetic and thermal laws.

2 The Necessary Condition for Dynamic Feedback Linearization

The set of C^∞ functions from a manifold X to a manifold Y is denoted by $C^\infty(X, Y)$. Consider the control system

$$\dot{x} = f(x, u) \tag{1}$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ and $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ such that $f(0, 0) = 0$.

Theorem 1 *Assume that $n > m \geq 0$ and that (1) is linearizable via dynamic feedback around the equilibrium point $(x, u) = (0, 0)$ in the sense of [3]. Then, there exist $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$, open neighborhoods of 0, such that*

- *the projection of the sub-manifold $\{(p, x, u) \in \mathbb{R}^n \times X \times U \mid p - f(x, u) = 0\}$ of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ onto $\mathbb{R}^n \times \mathbb{R}^n$ is a sub-manifold, Σ, of $\mathbb{R}^n \times \mathbb{R}^n$.*

- *Σ is a ruled sub-manifold: for each point $P \in \Sigma$, there exists an open segment of a straight line parallel to the p-coordinates and included in Σ.*

In [16] the same necessary condition is proposed. However, the proof given in [16], as explicitly stated, holds only up to feedback transformations, for the particular sub-class of dynamic compensators considered in [4]. We state here that this necessary condition is in fact general.

Proof: The proof is in the spirit of an old paper of Hilbert [11] where a closely related question is addressed for under-determinate differential systems. More precisely, the proof relies mainly on the notion of "integrallos Auflösung" due to Hilbert[11] (see equations (5)) and on the fundamental notion of linearizing output introduced by Martin [13]. We refer to [6, 7] for the differential-algebraic approach of Fliess and the proof of a similar necessary condition for flat systems.

The linearizability of (1) implies, according to [3], that the rank of $\frac{\partial f}{\partial u}(0,0)$ is maximum and equal to m. It implies also the existence of

1. a smooth dynamic compensator

$$\left\{ \begin{array}{rcl} \dot{z} &=& a(x,z,v) \\ u &=& b(x,z,v) \qquad (z \in Z \subset I\!\!R^q, \quad v \in V \subset I\!\!R^m) \end{array} \right. \tag{2}$$

   where, $q \geq 0$, Z and V are open neighborhoods of 0, $a(0,0,0) = 0$ and $b(0,0,0) = 0$;

2. a local C^∞ diffeomorphism

$$\left\{ \begin{array}{ccc} X \times Z & \rightarrow & \Xi \subset I\!\!R^{n+q} \\ (x,z) & \rightarrow & \xi = \phi(x,z) \end{array} \right. \tag{3}$$

   where $\phi(0,0) = 0$ and Ξ is an open neighborhood of 0;

such that (1) and (2) yield a $(n+q)$-dimensional dynamics,

$$\left\{ \begin{array}{rcl} \dot{x} &=& f(x, b(x,z,v)) \\ \dot{z} &=& a(x,z,v) \end{array} \right.$$

which becomes with respect to (3) a constant linear controllable system $\dot{\xi} = F\xi + Gv$. Up to static state feedback, this linear system may be written in Brunovsky canonical form [2],

$$\left\{ \begin{array}{rcl} y_1^{(\nu_1)} &=& v_1 \\ & \vdots & \\ y_m^{(\nu_m)} &=& v_m \end{array} \right.$$

where ν_1, ..., ν_m are the controllability indices and

$$\xi = (y_1, \ldots, y_1^{(\nu_1 - 1)}, \ldots, y_m, \ldots, y_m^{(\nu_m - 1)}).$$

The existence of the neighborhoods X et U (one can choose the same neighborhood X as the one used for the diffeomorphism ϕ) such that Σ is a sub-manifold results from the fact that the rank of f with respect to u is maximum around 0. The equations of Σ are obtained via the elimination of u from $p - f(x,u) = 0$: since the rank of f with respect to u around 0 is maximum, this elimination is always possible for $u \in U$ (implicit function theorem with a small neighborhood $U \subset \mathbb{R}^m$ of 0) and leads to $n-m$ equations

$$F(x,p) = 0 \tag{4}$$

where $F \in C^\infty(X \times P, \mathbb{R}^{n-m})$ (P is an open neighborhood of 0 in $\mathbb{R}^n$) defines a sub-manifold $\Sigma \subset X \times P$ (the rank of F with respect to p is maximum, i.e., $n-m$).

Denote by $A \in C^\infty(\Xi, X)$ the function corresponding to the components of ϕ^{-1} associated to x. Clearly, A is onto. Denote by α the largest integer such that

$$A_{y^{(\alpha)}} = \frac{\partial A}{\partial y^{(\alpha)}} \neq 0$$

($\alpha \leq \max((\nu_1 - 1), \ldots, (\nu_m - 1))$). For every smooth trajectory of (1), $t \to x(t) \in X$ and $t \to u(t) \in U$, there exists a smooth time function, $t \to y(t) \in \mathbb{R}^m$, such that $(y_1, \ldots, y_1^{(\nu_1-1)}, \ldots, y_m, \ldots, y_m^{(\nu_m-1)}) \in \Xi$ and

$$x(t) = A(y(t), \ldots, y^{(\alpha)}(t)) \tag{5}$$

(A is onto).

Conversely, every smooth function, $t \to y(t) \in \mathbb{R}^m$, such that

$$(y_1, \ldots, y_1^{(\nu_1-1)}, \quad \ldots \quad, y_m, \ldots, y_m^{(\nu_m-1)})$$

belongs to Ξ, leads to the smooth time function

$$t \longrightarrow x(t) = A(y(t), \ldots, y^{(\alpha)}(t))$$

which is a state trajectory for (1).

The substitution of (5) into (4) leads thus to an identity:

$$\begin{array}{ll} F(& A(\mathcal{Y}, \ldots, \mathcal{Y}^{(\alpha)}) \quad , \\ & A_{y^{(\alpha)}}(\mathcal{Y}, \ldots, \mathcal{Y}^{(\alpha)})\ \mathcal{Y}^{(\alpha+1)} + \ldots + A_y(\mathcal{Y}, \ldots, \mathcal{Y}^{(\alpha)})\ \dot{\mathcal{Y}} \quad) \equiv 0 \end{array} \tag{6}$$

for all $\mathcal{Y} \in \mathbb{R}^m$, ..., $\mathcal{Y}^{(\alpha)} \in \mathbb{R}^m$, $\mathcal{Y}^{(\alpha+1)} \in \mathbb{R}^m$ small enough.

Consider now $(x,p) \in \Sigma$. Since A is onto, there exists

$$(\overline{y}, \ldots, \overline{y}^{(\alpha)}) \in \mathbb{R}^m \times \ldots \times \mathbb{R}^m$$

close to 0 (not unique in general) such that $x = A(\overline{y}, \ldots, \overline{y}^{(\alpha)})$. The previous developments defining the surjection A show that, since $F(x,p) = 0$, there exists $\overline{y}^{(\alpha+1)} \in \mathbb{R}^m$ close to 0 such that

$$p = A_{y^{(\alpha)}} \overline{y}^{(\alpha+1)} + A_{y^{(\alpha-1)}} \overline{y}^{(\alpha)} + \ldots + A_y \dot{\overline{y}}$$

where the derivatives of A are evaluated at $(\overline{y}, \ldots, \overline{y}^{(\alpha)})$. Then (6) with $(\mathcal{Y}, \ldots, \mathcal{Y}^{(\alpha)}) = (\overline{y}, \ldots, \overline{y}^{(\alpha)})$ and $\mathcal{Y}^{(\alpha+1)} = \overline{y}^{(\alpha+1)} + \eta$ with $\eta \in \mathbb{R}^m$ arbitrary and small, implies that

$$F(x, p + A_{y^{(\alpha)}} \eta) = 0.$$

It suffices to take a small nonzero vector $a \in \mathbb{R}^n$ belonging to the image of the linear operator $A_{y^{(\alpha)}}(\overline{y}, \ldots, \overline{y}^{(\alpha)}) \neq 0$ to conclude that the segment $\{(x, p + \lambda a) |\ \lambda \in]-1, 1[\}$ belongs to Σ. ∎

3 Non-genericity of Dynamic Feedback Linearization

The fact that the sub-manifold Σ of Theorem 1 is ruled, is clearly non-generic. For the Whitney C^∞ topology on $C^\infty(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ [10, definition 3.1, page 42]), we have the following theorem.

Theorem 2 *Assume that $n > m \geq 1$. Denote by $C_0^\infty(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ the set of C^∞ mappings sending 0 to 0. The set of $f \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$, such that $\dot{x} = f(x,u)$ is not linearizable via dynamic feedback around the steady-state $(x,u) = 0$, contains an open dense subset for the Whitney C^∞ topology on $C_0^\infty(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$.*

The Whitney C^∞ topology is defined on $C_0^\infty(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ as the restriction to $C_0^\infty(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ of the standard Whitney C^∞ topology on $C^\infty(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$.

Proof: We can assume that the rank $\dfrac{\partial f}{\partial u}(0,0)$ is maximum and is equal to rank $\dfrac{\partial(f_1, \ldots, f_m)}{\partial u}(0,0)$. There exist small open neighborhoods of 0, $X \subset \mathbb{R}^n$, $U \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^n \times \mathbb{R}^m$, such that, the mapping

$$\left\{ \begin{array}{llll} \Phi: & \mathbb{R}^n \times \mathbb{R}^m & \longrightarrow & \mathbb{R}^n \times \mathbb{R}^m \\ & (x,u) & \longrightarrow & (x, f_1(x,u), \ldots, f_m(x,u)) \end{array} \right. \tag{7}$$

is a diffeomorphism from $X \times U$ to $W = \Phi(X \times U)$, sending 0 to 0. Moreover, as displayed on the figure here below, we can impose the following condition (by taking X and U small enough): there exists a compact neighborhood of 0, $K \subset W$, such that $\Phi^{-1}(K) \cap \partial(X \times U)$ is empty ($\partial(X \times U)$ denotes the boundary of $X \times U$ in $\mathbb{R}^n \times \mathbb{R}^m$).

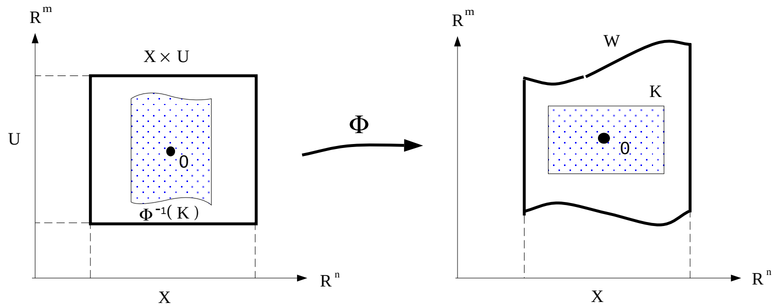

Figure 1:

Thus, for $(x, u) \in X \times U$, (1) is equivalent to

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & f_1(x,u) \\ & \vdots & \\ \dot{x}_m & = & f_m(x,u) \\ \dot{x}_{m+1} & = & F_{m+1}(x, \dot{x}_1, \ldots, \dot{x}_m) \\ & \vdots & \\ \dot{x}_n & = & F_n(x, \dot{x}_1, \ldots, \dot{x}_m) \end{array}\right.$$

where the C^∞ function $F = (F_{m+1}, \ldots, F_n)$ from W to $\mathbb{R}^{n-m}$ is derived from f and Φ by eliminating $(u_1, \ldots, u_m)$ from the $n-m$ last equations of (1). The equations

$$\left\{\begin{array}{rcl} p_{m+1} & = & F_{m+1}(x, p_1, \ldots, p_m) \\ & \vdots & \\ p_n & = & F_n(x, p_1, \ldots, p_m) \end{array}\right.$$

define the sub-manifold Σ of Theorem 1.

Consider now a C^∞ function H from W to $\mathbb{R}^{n-m}$ with a compact support included into K and with $H(0) = 0$. To H, we associate the C^∞ functions $F^\eta = F + \eta H$ for $\eta \in \mathbb{R}$. We associate also to F^η, f^η, a function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$, defined as follows:

– if $x \in X$ and $u \in U$ then

$$f^\eta(x,u) = \begin{pmatrix} f_1(x,u) \\ \vdots \\ f_m(x,u) \\ F^\eta_{m+1}(x, f_1(x,u), \ldots, f_m(x,u)) \\ \vdots \\ F^\eta_n(x, f_1(x,u), \ldots, f_m(x,u)) \end{pmatrix} \tag{8}$$

– otherwise $f^\eta(x,u) = f(x,u)$.

Clearly $f^\eta(0,0) = 0$. Moreover, f^η is C^∞. By construction of f^η, regularity problem can only occur on the boundary $\partial(X \times U)$. But, the assumptions, introduced for defining X, U and K with the mapping Φ of equation (7), show that f and f^η coincide on an open neighborhood of $\partial(X \times U)$. This implies the regularity of f^η around $\partial(X \times U)$.

To summarize, we have the following: to the C^∞ function $F^\eta = F + \eta H$, with H having a compact support included into K, there exits $f^\eta \in C^\infty_0(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ such that

– for the Whitney C^∞ topology, $\lim_{\eta \to 0} f^\eta = f$

– for $(x,u) \in X \times U$, the sub-manifold Σ^η of theorem 1 associated to f^η is given by the equations

$$\left\{ \begin{array}{rcl} p_{m+1} & = & F^\eta_{m+1}(x, p_1, \ldots, p_m) \\ & \vdots & \\ p_n & = & F^\eta_n(x, p_1, \ldots, p_m). \end{array} \right.$$

Assume now that (1) is linearizable via dynamic feedback around 0. Then, theorem 1 implies that for (x,p) small, there exists $a \in \mathbb{R}^n$, $a \neq 0$ and $\varepsilon > 0$, such that, for all $\lambda \in]-\varepsilon, \varepsilon[$:

$$\left\{ \begin{array}{rcl} p_{m+1} + \lambda a_{m+1} & = & F_{m+1}(x, p_1 + \lambda a_1, \ldots, p_m + \lambda a_m) \\ & \vdots & \\ p_n + \lambda a_n & = & F_n(x, p_1 + \lambda a_1, \ldots, p_m + \lambda a_m). \end{array} \right.$$

The derivation with respect to λ leads to, when $\lambda = 0$,

$$(a_{m+1}, \ldots, a_n) = \frac{\partial F}{\partial q} b$$

where $q = (p_1, \ldots, p_m)$ and $b = (a_1, \ldots, a_m)$. Since $a \neq 0$, the above relation implies that $b \neq 0$. Similarly, the second order derivation leads to

$$\frac{\partial^2 F}{\partial q^2}[b,b] = 0.$$

More generally, the derivation of order $k \in \{2, \ldots, m+1\}$ leads to

$$\frac{\partial^k F}{\partial q^k}[\underbrace{b, \ldots, b}_{k \text{ times}}] = 0. \tag{9}$$

This implies that the family of vectorial homogeneous polynomials of degree $k \in \{2, \ldots, m+1\}$

$$P_k(X_1, \ldots, X_m) = \frac{\partial^k F}{\partial q^k}\left[\underbrace{\begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix}, \ldots, \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix}}_{k \text{ times}}\right] = 0 \tag{10}$$

admits a non-zero common solution $X_1 = b_1, \ldots, X_m = b_m$.

Denote by $\mathcal{R}$ the resultant of the polynomials P_2, ..., P_{m+1}: $\mathcal{R}$ is an homogeneous vectorial polynomial in the coefficients of P_2, ..., P_{m+1}. Since

- the number of indeterminates X_i is equal to m,
- the number of scalar polynomials induced by P_k is equal to $m(n-m) \geq m$ since $n > m$,
- each P_k is homogeneous of degree k and its coefficients are arbitrary since they correspond to the k-th derivatives of F with respect to $(p_1, \ldots, p_m)$,

$\mathcal{R}$ is a nonzero vectorial polynomial of the coefficients of P_2, ..., P_{m+1}.

The elimination theory [20, chapter XI] says that, since the homogeneous polynomials $(P_k)_{2 \leq k \leq m+1}$ admit a common nonzero root, their coefficients are a root of the nonzero resultant $\mathcal{R}$. This means that, for all $(x, q) \in W$, F satisfies the following partial differential relations:

$$\mathcal{R}\left[\left(\frac{\partial^{i_1+\ldots+i_m} F_j}{\partial p_1^{i_1} \ldots \partial p_m^{i_m}}\right)_{\substack{m+1 \leq j \leq n \\ i_1 \geq 0, \ldots, i_m \geq 0 \\ 2 \leq i_1 + \ldots + i_m \leq m+1}}\right] = 0. \tag{11}$$

It is clear that there exists $H \in C_0^\infty(W, \mathbb{R}^{n-m})$, with a support contained in K, such that for $|\eta|$ small enough, the vectorial function of

$(x, q) \in W$,

$$\mathcal{R}\left[\left(\frac{\partial^{i_1+\ldots+i_m} F_j^\eta}{\partial p_1^{i_1}\ldots\partial p_m^{i_m}}\right)\begin{array}{l} m+1\leq j\leq n \\ i_1\geq 0,\ldots,i_m\geq 0 \\ 2\leq i_1+\ldots+i_m\leq m+1\end{array}\right]$$

is not equal to 0 for $(x, q) = 0$. This implies that $f^\eta \in C^\infty(I\!R^n \times I\!R^m, I\!R^n)$ associated to $F^\eta = F + \eta H$ (see (8)) leads to a control system $\dot{x} = f^\eta(x, u)$ that is not linearizable via dynamic feedback. Since $\lim_{\eta\to 0} f^\eta = f$ for the Whitney C^∞ topology, the density of nonlinearizable systems that do not satisfy (11) is proved.

Assume now that $f \in C_0^\infty(I\!R^n \times I\!R^m, I\!R^n)$ leads to $F \in C^\infty(W, I\!R^{n-m})$ that does not satisfy (11). Then, it is obvious that every g close to f for the Whitney C^∞ topology leads to $G \in C_0^\infty(W, I\!R^{n-m})$ that does not satisfy (11). This results from the following facts:

- the mapping that sends f to $F \in C_0^\infty(W, I\!R^{n-m})$, as stated above, can be well defined for a small neighborhood of f in $C_0^\infty(I\!R^n \times I\!R^m, I\!R^n)$;
- this mapping is continuous for the Whitney C^∞ topology. This results from the fact that the derivatives of F with respect to x and q are obtained via polynomial expressions of the derivatives of f with respect to x and u and of $\left[\frac{\partial(f_1,\ldots,f_m)}{\partial u}\right]^{-1}$;
- the mapping that associates to F the vectorial function defined by the left-hand side of (11) is continuous for the C^∞ topology.

We have proved that the set of $f \in C_0^\infty(I\!R^n \times I\!R^m, I\!R^n)$ such that the rank of $\frac{\partial f}{\partial u}(0,0)$ is maximum and leading, locally around 0, to mappings F that do not satisfy (11) is open and dense. Since such mappings f lead, according to Theorem 1, to non linearizable control systems, the proof is finished. ■

Remark 1 *The necessary condition of Theorem 1 is always satisfied for affine control systems (i.e. when f is an affine function of u): as shown in [18], this case seems to be much more complicated. For example, it is well known [3, 18] that generic affine control systems with $n = m + 1$ are linearizable via dynamic feedback. Theorem 2 says that this is no more true when $n = m + 1$ and f is not supposed to be an affine function of u.*

4 Discussion

The mathematical notion of genericity and structural stability corresponds to an idealization and formalization of the fact that every modeling process is always an approximation: for the dynamic model, $\dot{x} = f(x, u)$, f is not precisely known. Theorem 2 means that linearizability via dynamic feedback for arbitrary smooth systems $\dot{x} = f(x, u)$ is not a structurally stable property when the Whitney C^∞ topology is used. More precisely, the opposite property is generic (in the sense given in [10, page 141]). Notice that our result does not imply that the non-linearizability via dynamic feedback is a structurally stable property, although this might be true.

According to the "structural stability dogma", i.e., the properties relevant to applications are only the structurally stable ones [19], control models linearizable via dynamic feedback must be considered as irrelevant to applications. In fact, the situation is not as simple as it seems. Many dynamical processes, reasonably modeled, are linearizable via dynamic feedback (see, e.g., [1, 9, 6] for mechanical systems and [14] for chemical reactors)..

In the appendix, we give a new practical and nontrivial linearizable example: the chemical polymerization reactor with two controls considered in [17][1]. This system is linearizable via dynamic feedback for generic kinetic and thermal laws (functions R_m, k_i, ϕ and f_6 of (12)). This property is important since it takes into account the usual fact that the kinetic and thermal laws are semi-empirical laws elaborated from real data. For this particular system, it is thus natural to consider only structural control properties that do not depend on the special form of these laws.

It results that dynamic feedback linearizability makes sense for this chemical reactor. On the opposite, the Whitney C^∞ topology, used for representing model uncertainties, reveals to be too coarse for this particular system. Some a priori knowledge on the system structure must be taken into account in the definition of a specific topology and admissible perturbations of the model equations. It is reasonable here to define such admissible perturbations as arbitrary smooth perturbations of the functions R_m, k_i, ϕ and f_6.

Acknowledgments

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[1] The author thanks J. Mac Gregor for this reference.

A A Polymerization Reactor Linearizable via Dynamic Feedback

Consider the chemical reactor studied in [17]. The equations of the reactor are (the notations are those of [17])

$$
\begin{aligned}
\dot{C}_m &= \frac{C_{m_{m_s}}}{\tau} - \left(1 + \bar{\varepsilon}\frac{\mu_1}{\mu_1 + M_m C_m}\right)\frac{C_m}{\tau} + R_m(C_m, C_i, C_s, T) \\
\dot{C}_i &= -k_i(T)C_i + u_2\frac{C_{i_{i_s}}}{V} - \left(1 + \bar{\varepsilon}\frac{\mu_1}{\mu_1 + M_m C_m}\right)\frac{C_i}{\tau} \\
\dot{C}_s &= u_2\frac{C_{s_{i_s}}}{V} + \frac{C_{s_{m_s}}}{\tau} - \left(1 + \bar{\varepsilon}\frac{\mu_1}{\mu_1 + M_m C_m}\right)\frac{C_s}{\tau} \\
\dot{\mu}_1 &= -M_m R_m(C_m, C_i, C_s, T) - \left(1 + \bar{\varepsilon}\frac{\mu_1}{\mu_1 + M_m C_m}\right)\frac{\mu_1}{\tau} \\
\dot{T} &= \phi(C_m, C_i, C_s, \mu_1, T) + \alpha_1 T_j \\
\dot{T}_j &= f_6(T, T_j) + \alpha_4 u_1
\end{aligned}
\tag{12}
$$

where

- $x = (C_m, C_i, C_s, \mu_1, T, T_j)$ is the state.
- $u = (u_1, u_2)$ is the control
- $p = (C_{m_{m_s}}, M_m, \bar{\varepsilon}, \tau, C_{i_{i_s}}, C_{s_{m_s}}, C_{s_{i_s}}, V, \alpha_1, \alpha_4)$ are constant positive physical parameters.
- the functions R_m, k_i, ϕ and f_6 can be considered arbitrary since they are not precisely determined: they involve kinetic laws, heat transfer coefficients and reaction enthalpies; their expressions are derived from real data and semi-empirical considerations.

Symbolically, (12) is denoted by $\dot{x} = f(x, u, p)$.

Consider now the output $y = (y_1, y_2) = (C_{s_{i_s}} C_i - C_{i_{i_s}} C_s, M_m C_m + \mu_1)$. We shall see that y is a linearizing output [6] of (12) for generic functions R_m, k_i, ϕ and f_6. In other words, we will prove that the entire state x and the control u are functions of y and a finite number of its derivatives. This means that (12) is linearizable via endogenous dynamic feedback [13] around equilibria for generic functions R_m, k_i, ϕ and f_6 (the term generic is related here to the Whitney topology for C^∞ mappings).

We just prove now in details that x and u are functions of y and a finite number of its derivatives, for R_m, k_i, ϕ and f_6 generic. We do not prove in details, although this is possible but has little interest for our purpose, the fact that, generically for R_m, k_i, ϕ and f_6, (12) is linearizable via dynamic feedback around an equilibrium in the sense of [3].

We have

$$\dot{y}_2 = M_m \frac{C_{m_{m_s}}}{\tau} - \left(1 + \overline{\varepsilon}\frac{\mu_1}{y_2}\right)\frac{y_2}{\tau}.$$

Thus μ_1 is a function of y_2 and $\dot{y}_2$. Since $y_2 = M_m C_m + \mu_1$, C_m is also a function of y_2 and $\dot{y}_2$. We have

$$\begin{cases} \dot{y}_1 & = & -C_{s_{i_s}} k_i(T) C_i - \dfrac{C_{s_{m_s}} C_{i_{i_s}}}{\tau} - \left(1 + \overline{\varepsilon}\dfrac{\mu_1}{y_2}\right)\dfrac{y_1}{\tau} \\ \dot{\mu}_1 & = & -M_m R_m(C_m, C_i, C_s, T) - \left(1 + \overline{\varepsilon}\dfrac{\mu_1}{y_2}\right)\dfrac{\mu_1}{\tau} \\ y_1 & = & C_{s_{i_s}} C_i - C_{i_{i_s}} C_s. \end{cases}$$

Thus (C_i, C_s, T) is a function of $(y_1, \dot{y}_1, y_2, \mu_1, \dot{\mu}_1, C_m)$. According to the previous computations, (C_i, C_s, T) is a function of $(y_1, \dot{y}_1, y_2, \dot{y}_2, \ddot{y}_2)$.

Similarly, since

$$\begin{cases} \dot{C}_s & = & u_2 \dfrac{C_{s_{i_s}}}{V} + \dfrac{C_{s_{m_s}}}{\tau} - \left(1 + \overline{\varepsilon}\dfrac{\mu_1}{\mu_1 + M_m C_m}\right)\dfrac{C_s}{\tau} \\ \dot{T} & = & \phi(C_m, C_i, C_s, \mu_1, T) + \alpha_1 T_j, \end{cases}$$

(T_j, u_2) is a function of $(y_1, \dot{y}_1, \ddot{y}_1, y_2, \dot{y}_2, \ddot{y}_2, y_2^{(3)})$. Since

$$\dot{T}_j = f_6(T, T_j) + \alpha_4 u_1$$

u_1 is a function of $(y_1, \dot{y}_1, \ddot{y}_1, y_1^{(3)}, y_2, \dot{y}_2, \ddot{y}_2, y_2^{(3)}, y_2^{(4)})$.

Thus x is a function of

$$(y_1, \dot{y}_1, \ddot{y}_1, y_2, \dot{y}_2, \ddot{y}_2, y_2^{(3)})$$

and u depends on

$$(y_1, \dot{y}_1, \ddot{y}_1, y_1^{(3)}, y_2, \dot{y}_2, \ddot{y}_2, y_2^{(3)}, y_2^{(4)}).$$

These calculations show also that, generically, the extended system

$$\begin{cases} \dot{x} & = & f(x, u_1, u_2, p) \\ \dot{u}_2 & = & \tilde{u}_2 \end{cases}$$

with the control $(u_1, \tilde{u}_2)$ is linearizable via static feedback [12] and can be transformed into

$$y_1^{(3)} = v_1, \quad y_2^{(4)} = v_2.$$

The reader can verify that (12) is not linearizable via static feedback. Notice furthermore that the linearizing output admits a clear physical interpretation:

– $y_1 = \begin{vmatrix} C_{s_{i_s}} & C_s \\ C_{i_{i_s}} & C_i \end{vmatrix}$ measures the degree of colinearity of the solvent and initiator compositions between the inlet initiator stream and the reactor.

– y_2 is the sum of the concentration of monomer and polymer in the reactor.

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Centre Automatique et Systèmes, École des Mines de Paris, 60, Bd Saint-Michel, 75272 Paris Cedex 06, France

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